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A stochastic maximum principle for processes driven by fractional Brownian motion

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Abstract

We prove a stochastic maximum principle for controlled processes $X(t)=X^{(u)}(t)$ of the form

$$\mathrm{d}X(t)=b(t,X(t),u(t))\,\mathrm{d}t+\sigma(t,X(t),u(t))\,\mathrm{d}B^{(H)}(t),$$

where $B^{(H)}(t)$ is m-dimensional fractional Brownian motion with Hurst parameter $H=(H_1,\dots,H_m)\in(\frac{1}{2},1)^m$. As an application we solve a problem about minimal variance hedging in an incomplete market driven by fractional Brownian motion. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $H=(H_1,\dots,H_m)$ with $\frac{1}{2}<H_j<1$, $j=1,2,\dots,m$, and let $B^{(H)}(t)=(B_1^{(H)}(t),\dots,B_m^{(H)}(t))$, $t\in\mathbb{R}$ be m-dimensional fractional Brownian motion, i.e. $B^{(H)}(t)=B^{(H)}(t,\omega)$,

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$(t,\omega)\in\mathbb{R}\times\Omega$ is a Gaussian process in $\mathbb{R}^m$ such that

$$\mathbb{E}[B^{(H)}(t)] = B^{(H)}(0) = 0 \tag{1.1}$$

and

$$\mathbb{E}[B_j^{(H)}(s)B_k^{(H)}(t)] = \tfrac{1}{2}\{|s|^{2H_j} + |t|^{2H_j} - |t-s|^{2H_j}\}\delta_{jk},$$
$$1 \leqslant j,k \leqslant n, \quad s,t\in\mathbb{R}, \tag{1.2}$$

where

$$\delta_{jk} = \begin{cases} 0 & \text{when } j\neq k, \\ 1 & \text{when } j=k. \end{cases}$$

Here $\mathbb{E}=\mathbb{E}_\mu$ denotes the expectation with respect to the probability law $\mu=\mu_H$ for $B^{(H)}(\cdot)$. This means that the components $B_1^{(H)}(\cdot),\dots,B_m^{(H)}(\cdot)$ of $B^{(H)}(\cdot)$ are m independent one-dimensional fractional Brownian motions with Hurst parameters $H_1,H_2,\dots,H_m$, respectively. We refer to Mandelbrot and Van Ness (1968), Norros et al. (1999) and Shiryaev (1999) for more information about fractional Brownian motion. Because of its interesting properties (e.g. long range dependence and self-similarity of the components) $B^{(H)}(t)$ has been suggested as a replacement of *standard Brownian motion* $B(t)$ (corresponding to $H_j=\frac{1}{2}$ for all $j=1,\dots,m$) in several stochastic models, including finance.

Unfortunately, $B^{(H)}(\cdot)$ is neither a semimartingale nor a Markov process, so the powerful tools from the theories of such processes are not applicable when studying $B^{(H)}(\cdot)$. Nevertheless, an efficient stochastic calculus of $B^{(H)}(\cdot)$ can be developed. This calculus uses an Itô type of integration with respect to $B^{(H)}(\cdot)$ and white noise theory. See Duncan et al. (2000), Hu and Øksendal (1999), and Elliott and van der Hoek (2000) for details. For applications to finance see Hu and Øksendal (1999), Hu et al. (2000a, 2000b). In Hu (2000, 2001), Hu et al. (2000) and Øksendal and Zhang (2001) the theory is extended to multi-parameter fractional Brownian fields $B^{(H)}(x)$; $x\in\mathbb{R}^{\mathrm{d}}$ and applied to stochastic partial differential equations driven by such fractional white noise.

The purpose of this paper is to establish a stochastic maximum principle for stochastic control of processes driven by $B^{(H)}(\cdot)$. We illustrate the result by applying it to a problem about minimal variance hedging in finance.

2. Preliminaries

For the convenience of the reader we recall here some of the basic results of fractional Brownian motion calculus. Let $B^{(H)}(t)$ be one-dimensional in the following.

Define, for given $H\in(\frac{1}{2},1)$,

$$\phi(s,t) = \phi_H(s,t) = H(2H-1)|s-t|^{2H-2}, \quad s,t\in\mathbb{R}. \tag{2.1}$$

As in Hu and Øksendal (1999) we will assume that Ω is the space $\mathscr{S}'(\mathbb{R})$ of tempered distributions on $\mathbb{R}$, which is the dual of the Schwartz space $\mathscr{S}(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$. If $\omega \in \mathscr{S}'(\mathbb{R})$ and $f \in \mathscr{S}(\mathbb{R})$ we let $\langle \omega, f \rangle = \omega(g)$ denote the action of ω applied to f. It can be extended to all $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_\phi^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s,t) \, \mathrm{d}s \, \mathrm{d}t < \infty.$$

The space of all such functions f is denoted by $L_\phi^2(\mathbb{R})$.

If $F : \Omega \to \mathbb{R}$ is a given function we let

$$D_t^\phi F = \int_{\mathbb{R}} D_r F \, \phi(r,t) \, \mathrm{d}r \tag{2.2}$$

denote the Malliavin ϕ-derivative of F at t (if it exists) (see [Duncan et al. (2000), Definition 3.4]). Define $\mathscr{L}_\phi^{1,2}$ to be the set of processes $g(t,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $D_s^\phi g(s)$ exists for a.a. $s \in \mathbb{R}$ and

$$\|g\|_{\mathscr{L}_\phi^{1,2}}^2 := \mathbb{E}\left[\int_{\mathbb{R}} \int_{\mathbb{R}} g(s) g(t) \phi(s,t) \, \mathrm{d}s \, \mathrm{d}t + \left(\int_{\mathbb{R}} D_s^\phi g(s) \, \mathrm{d}s \right)^2 \right] < \infty. \tag{2.3}$$

We let $\int_{\mathbb{R}} \sigma(t,\omega) \, \mathrm{d}B^{(H)}(t)$ denote the *fractional Itô-integral* of the process $\sigma(t,\omega)$ with respect to $B^{(H)}(t)$, as defined in Duncan et al. (2000). In particular, this means that if σ belongs to the family $\mathbb{S}$ of step functions of the form

$$\sigma(t,\omega) = \sum_{i=1}^N \sigma_i(\omega) \chi_{[t_i, t_{i+1})}(t), \quad (t,\omega) \in \mathbb{R} \times \Omega,$$

where $0 \leqslant t_1 < t_2 < \cdots < t_{N+1}$, then

$$\int_{\mathbb{R}} \sigma(t,\omega) \, \mathrm{d}B^{(H)}(t) = \sum_{i=1}^N \sigma_i(\omega) \diamond (B^{(H)}(t_{i+1}) - B^{(H)}(t_i)), \tag{2.4}$$

where $\diamond$ denotes the Wick product. For $\sigma(t) = \sigma(t,\omega) \in \mathbb{S} \cap \mathscr{L}_\phi^{1,2}$ we have the isometry

$$\begin{aligned} \mathbb{E}\left[\int_{\mathbb{R}} \sigma(t,\omega) \, \mathrm{d}B^{(H)}(t) \right]^2 &= \mathbb{E}\left[\int_{\mathbb{R}^2} \sigma(s) \sigma(t) \phi(s,t) \, \mathrm{d}s \, \mathrm{d}t + \left(\int_{\mathbb{R}} D_s^\phi \sigma(s) \, \mathrm{d}s \right)^2 \right] \\ &= \|\sigma\|_{\mathscr{L}_\phi^{1,2}}^2, \end{aligned} \tag{2.5}$$

where $\mathbb{E} = \mathbb{E}_{\mu_H}$. Using this we can extend the integral $\int_{\mathbb{R}} \sigma(t,\omega) \, \mathrm{d}B^{(H)}(t)$ to $\mathscr{L}_\phi^{1,2}$. Note that if $\sigma, \theta \in \mathscr{L}_\phi^{1,2}$, we have, by polarization

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathbb{R}} \sigma(t,\omega) \, \mathrm{d}B^{(H)}(t) \int_{\mathbb{R}} \theta(t,\omega) \, \mathrm{d}B^{(H)}(t) \right] \\ &\quad = \mathbb{E}\left[\int_{\mathbb{R}^2} \sigma(s) \theta(t) \phi(s,t) \, \mathrm{d}s \, \mathrm{d}t + \int_{\mathbb{R}} D_s^\phi \sigma(s) \, \mathrm{d}s \int_{\mathbb{R}} D_t^\phi \theta(t) \, \mathrm{d}t \right]. \end{aligned} \tag{2.6}$$

Also note that we need not assume that the integrand $\sigma \in \mathscr{L}_\phi^{1,2}$ is adapted to the filtration $\mathscr{F}_t^{(H)}$ generated by $B^{(H)}(s,\cdot)$; $s \leqslant t$.

An important property of this fractional Itô-integral is that

$$\mathbb{E}\left[\int_{\mathbb{R}} \sigma(t,\omega)\,\mathrm{d}B^{(H)}(t)\right] = 0 \quad \text{for all } \sigma \in \mathscr{L}_\phi^{1,2}. \tag{2.7}$$

(see [Duncan et al. (2000), Theorem 3.9]).

We give three versions of the fractional Itô formula, in increasing order of complexity.

Theorem 2.1 (Duncan et al. (2000), Theorem 4.1). *Let $f \in C^2(\mathbb{R})$ with bounded second order derivatives. Then for $t \geqslant 0$*

$$f(B^{(H)}(t)) = f(B^{(H)}(0)) + \int_0^t f'(B^{(H)}(s))\,\mathrm{d}B^{(H)}(s) + H\int_0^t s^{2H-1} f''(B^{(H)}(s))\,\mathrm{d}s. \tag{2.8}$$

Theorem 2.2 (Duncan et al. (2000), Theorem 4.3). *Let $X(t) = \int_0^t \sigma(s,\omega)\,\mathrm{d}B^{(H)}(s)$, where $\sigma \in \mathscr{L}_\phi^{1,2}$ and assume $f \in C^2(\mathbb{R}_+ \times \mathbb{R})$ with bounded second order derivatives. Then for $t \geqslant 0$*

$$f(t,X(t)) = f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,X(s))\,\mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s,X(s))\sigma(s)\,\mathrm{d}B^{(H)}(s) + \int_0^t \frac{\partial^2 f}{\partial x^2}(s,X(s))\sigma(s)D_s^\phi X(s)\,\mathrm{d}s. \tag{2.9}$$

Finally we give an m-dimensional version.

Let $B^{(H)}(t) = (B_1^{(H)}(t),\ldots,B_m^{(H)}(t))$ be an m-dimensional fractional Brownian motion with Hurst parameter $H = (H_1,\ldots,H_m) \in (\frac{1}{2},1)^m$, as in Section 1. Since we are here dealing with m independent fractional Brownian motions we may regard Ω as the product of m independent copies of $\bar{\Omega}$ and write $\omega = (\omega_1,\ldots,\omega_m)$ for $\omega \in \Omega$. Then in the following the notation $D_{k,s}^\phi Y$ means the Malliavin ϕ-derivative with respect to ω_k and could also be written

$$D_{k,s}^\phi Y = \int_{\mathbb{R}} \phi_{H_k}(s,t) D_{k,t} Y\,\mathrm{d}t = \int_{\mathbb{R}} \phi_{H_k}(s,t)\frac{\partial Y}{\partial \omega_k}(t,\omega)\,\mathrm{d}t. \tag{2.10}$$

Similar to the one-dimensional case discussed in Section 1, we can define the multi-dimensional fractional (Wick–Itô) integral

$$\int_{\mathbb{R}} f(t,\omega)\,\mathrm{d}B^{(H)}(t) = \sum_{j=1}^m \int_{\mathbb{R}} f_j(t,\omega)\,\mathrm{d}B_j^{(H)}(t) \in L^2(\mu) \tag{2.11}$$

for all processes $f(t,\omega)=(f_1(t,\omega),\ldots,f_m(t,\omega))\in\mathbb{R}^m$ such that, for all $j=1,2,\ldots,m$,

$$\|f_j\|^2_{\mathscr{L}^{1,2}_{\phi_j}} := \mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} f_j(s)f_j(t)\phi_j(s,t)\,\mathrm{d}s\,\mathrm{d}t + \left(\int_{\mathbb{R}} D^{\phi_j}_{j,t} f_j(t)\,\mathrm{d}t\right)^2\right] < \infty, \quad (2.12)$$

where $\phi_j = \phi_{H_j}$; $1 \leqslant j \leqslant m$.

Denote the set of all such m-dimensional processes f by $\mathscr{L}^{1,2}_{\phi}(m)$, where $\phi = (\phi_1,\ldots,\phi_m)$.

It can be proved (see Biagini and Øksendal (2001)) that for $f,g\in\mathscr{L}^{1,2}_{\phi}(m)$ we have the following fractional multi-dimensional Itô isometry

$$\begin{aligned}\mathbb{E}&\left[\left(\int_{\mathbb{R}} f\,\mathrm{d}B^{(H)}\right)\left(\int_{\mathbb{R}} g\,\mathrm{d}B^{(H)}\right)\right]\\ &= \mathbb{E}\left[\sum_{i=1}^m \int_{\mathbb{R}}\int_{\mathbb{R}} f_i(s)g_i(t)\phi_i(s,t)\,\mathrm{d}s\,\mathrm{d}t\right.\\ &\qquad\left. + \sum_{i,j=1}^m \left(\int_{\mathbb{R}} D^{\phi}_{j,t} f_i(t)\,\mathrm{d}t\right)\left(\int_{\mathbb{R}} D^{\phi}_{i,t} g_j(t)\,\mathrm{d}t\right)\right]. \end{aligned}\quad (2.13)$$

We put

$$\begin{aligned}(f,g)_{\mathbb{L}^{1,2}_{\phi}(m)} = \mathbb{E}&\left[\sum_{i=1}^m \int_{\mathbb{R}}\int_{\mathbb{R}} f_i(s)g_i(t)\phi_i(s,t)\,\mathrm{d}s\,\mathrm{d}t\right.\\ &\qquad\left. + \sum_{i,j=1}^m \left(\int_{\mathbb{R}} D^{\phi}_{j,t} f_i(t)\,\mathrm{d}t\right)\left(\int_{\mathbb{R}} D^{\phi}_{i,t} g_j(t)\,\mathrm{d}t\right)\right] \end{aligned}\quad (2.14)$$

and define

$$\mathbb{L}^{1,2}_{\phi}(m) = \{f\in\mathscr{L}^{1,2}_{\phi}(m);\, \|f\|^2_{\mathbb{L}^{1,2}_{\phi}(m)} := (f,f)_{\mathbb{L}^{1,2}_{\phi}(m)} < \infty\}.$$

Now suppose $\sigma_i \in \mathscr{L}^{1,2}_{\phi}(m)$ for $1 \leqslant i \leqslant n$. Then we can define $X(t)=(X_1(t),\ldots,X_n(t))$ where

$$X_i(t,\omega) = \sum_{j=1}^m \int_0^t \sigma_{ij}(s,\omega)\,\mathrm{d}B^{(H)}_j(s); \quad 1 \leqslant i \leqslant n. \quad (2.15)$$

We have the following multi-dimensional fractional Itô formula.

Theorem 2.3. *Let $f\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^n)$ with bounded second order derivatives. Then, for $t \geqslant 0$,*

$$\begin{aligned}f(t,X(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,X(s))\,\mathrm{d}s + \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s,X(s))\,\mathrm{d}X_i(s)\\ &\quad + \int_0^t \left\{\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X(s)) \sum_{k=1}^m \sigma_{ik}(s) D^{\phi}_{k,s}(X_j(s))\right\}\mathrm{d}s \end{aligned}\quad (2.16)$$

$$= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))\,\mathrm{d}s$$

$$+ \sum_{j=1}^{m} \int_0^t \left[\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(s, X(s))\sigma_{ij}(s,\omega) \right] \mathrm{d}B_j^{(H)}(s)$$

$$+ \int_0^t \mathrm{Tr}[\Lambda^{\mathrm{T}}(s) f_{xx}(s, X(s))]\,\mathrm{d}s. \tag{2.17}$$

Here $\Lambda = [\Lambda_{ij}] \in \mathbb{R}^{n\times m}$ *with*

$$\Lambda_{ij}(s) = \sum_{k=1}^{m} \sigma_{ik} D_{k,s}^{\phi}(X_j(s)), \quad 1 \leqslant i \leqslant n, \; 1 \leqslant j \leqslant m, \tag{2.18}$$

$$f_{xx} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{1 \leqslant i,j \leqslant n} \tag{2.19}$$

and $(\cdot)^{\mathrm{T}}$ *denotes matrix transposed and* $\mathrm{Tr}[\,\cdot\,]$ *denotes matrix trace.*

The following useful result is a multi-dimensional version of Theorem 4.2 in (Duncan et al., 2000).

Theorem 2.4. *Let*

$$X(t) = \sum_{j=1}^{m} \int_0^t \sigma_j(r,\omega)\,\mathrm{d}B_j^{(H)}(r), \quad \sigma = (\sigma_1, \ldots, \sigma_m) \in \mathscr{L}_{\phi}^{1,2}(m). \tag{2.20}$$

Then

$$D_{k,s}^{\phi} X(t) = \sum_{j=1}^{m} \int_0^t D_{k,s}^{\phi} \sigma_j(r)\,\mathrm{d}B_j^{(H)}(r) + \int_0^t \sigma_k(r)\phi_{H_k}(s,r)\,\mathrm{d}r, \quad 1 \leqslant k \leqslant m. \tag{2.21}$$

In particular, if $\sigma_j(r)$ *is deterministic for all* $j \in \{1, 2, \ldots, m\}$ *then*

$$D_{k,s}^{\phi} X(t) = \int_0^t \sigma_k(r)\phi_{H_k}(s,r)\,\mathrm{d}r. \tag{2.22}$$

Now we have the following integration by parts formula.

Corollary 2.5. *Let* $X(t)$ *and* $Y(t)$ *be two processes of the form*

$$\mathrm{d}X(t) = \mu(t,\omega)\mathrm{d}t + \sigma(t,\omega)\,\mathrm{d}B^{(H)}(t), \quad X(0) = x \in \mathbb{R}^n$$

and

$$\mathrm{d}Y(t) = \nu(t,\omega)\mathrm{d}t + \theta(t,\omega)\,\mathrm{d}B^{(H)}(t), \quad Y(0) = y \in \mathbb{R}^n,$$

where $\mu:\mathbb{R}\times\Omega\to\mathbb{R}^n$, $v:\mathbb{R}\times\Omega\to\mathbb{R}^n$, $\sigma:\mathbb{R}\times\Omega\to\mathbb{R}^{n\times m}$ *and* $\theta:\mathbb{R}\times\Omega\to\mathbb{R}^{n\times m}$ *are given processes with rows* σ_i, $\theta_i\in\mathscr{L}_\phi^{1,2}(m)$ *for* $1\leqslant i\leqslant n$ *and* $B^H(\cdot)$ *is an m-dimensional fractional Brownian motion.*

(a) *Then, for* $T>0$,

$$\mathbb{E}[X(T)Y(T)]=xy+\mathbb{E}\left[\int_0^T X(s)\,\mathrm{d}Y(s)\right]+\mathbb{E}\left[\int_0^T Y(s)\,\mathrm{d}X(s)\right]$$
$$+\mathbb{E}\left[\int_0^T\int_0^T\sum_{i=1}^n\sum_{k=1}^m\sigma_{ik}(s)\theta_{ik}(t)\phi_{H_k}(s,t)\,\mathrm{d}s\,\mathrm{d}t\right]$$
$$+\mathbb{E}\left[\sum_{i=1}^n\sum_{j,k=1}^m\left(\int_{\mathbb{R}}D_{j,t}^\phi\sigma_{ik}(t)\,\mathrm{d}t\right)\left(\int_{\mathbb{R}}D_{k,t}^\phi\theta_{ij}(t)\,\mathrm{d}t\right)\right] \qquad (2.23)$$

provided that the first two integrals exist.

(b) *In particular, if* $\sigma(\cdot)$ *or* $\theta(\cdot)$ *is deterministic then*

$$\mathbb{E}[X(T)Y(T)]=xy+\mathbb{E}\left[\int_0^T X(s)\,\mathrm{d}Y(s)\right]+\mathbb{E}\left[\int_0^T Y(s)\,\mathrm{d}X(s)\right]$$
$$+\mathbb{E}\left[\int_0^T\int_0^T\sum_{i=1}^n\sum_{k=1}^m\sigma_{ik}(s)\theta_{ik}(t)\phi_{H_k}(s,t)\,\mathrm{d}s\,\mathrm{d}t\right]. \qquad (2.24)$$

Proof. This follows from Theorem 2.3 applied to the function $f(t,x,y)=xy$, combined with (2.13). □

3. Stochastic differential equations

For given functions $b:\mathbb{R}\times\mathbb{R}\times\Omega\to\mathbb{R}$ and $\sigma:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ consider the stochastic differential equation

$$\mathrm{d}X(t)=b(t,X(t))\,\mathrm{d}t+\sigma(t,X(t))\,\mathrm{d}B^{(H)}(t),\quad t\in[0,T], \qquad (3.1)$$

where the initial value $X(0)\in L^2(\mu_H)$ or the terminal value $X(T)\in L^2(\mu_H)$ is given. The Itô isometry for the stochastic integral becomes

$$\mathbb{E}\left(\int_0^T\sigma(t,X(t))\,\mathrm{d}B^{(H)}(t)\right)^2=\mathbb{E}\left(\int_0^T\int_0^T\sigma(t,X(t))\sigma(s,X(s))\phi(s,t)\,\mathrm{d}s\,\mathrm{d}t\right)$$
$$+\mathbb{E}\left\{\left(\int_0^{\mathrm{T}}\sigma_x'(s,X(s))D_s^\phi X(s)\,\mathrm{d}s\right)^2\right\}. \qquad (3.2)$$

Because of the appearance of the term $D_s^\phi X(s)$ on the right-hand side of the above identity, we may not directly apply the Picard iteration to solve (3.1).

In this section, we will solve the following quasi-linear stochastic differential equations using the theory developed in Hu and Øksendal (1996, 1999):

$$\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + (\sigma_t X(t) + a_t)\,\mathrm{d}B^{(H)}(t), \tag{3.3}$$

where σ_t and a_t are given deterministic functions, $b(t,x) = b(t,x,\omega)$ is (almost surely) continuous with respect to t and x and globally Lipschitz continuous on x, the initial condition $X(0)$ or the terminal condition $X(T)$ is given. For simplicity we will discuss the case when $a_t = 0$ for all $t \in [0,T]$. Namely, we shall consider

$$\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sigma_t X(t)\,\mathrm{d}B^{(H)}(t). \tag{3.4}$$

We need the following result, which is a fractional version of Gjessing's lemma (see e.g. Theorem 2.10.7 in Holden et al., 1996).

Lemma 3.1. *Let $G \in L^2(\mu_H)$ and*

$$F = \exp^{\diamond}\left(\int_{\mathbb{R}} f(t)\,\mathrm{d}B^{(H)}(t)\right) = \exp\left(\int_{\mathbb{R}} f(t)\,\mathrm{d}B^{(H)}(t) - \frac{1}{2}\|f\|_\phi^2\right),$$

where f is deterministic and such that

$$\|f\|_\phi^2 := \int_{\mathbb{R}^2} f(s)f(t)\phi(s,t)\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

Then

$$F \diamond G = F\tau_{\hat{f}} G, \tag{3.5}$$

where $\diamond$ is the Wick product defined in Hu and Øksendal (1999), *$\hat{f}$ is given by*

$$\int_{\mathbb{R}^2} f(s)g(t)\phi(s,t)\,\mathrm{d}s\,\mathrm{d}t = \int_{\mathbb{R}} \hat{f}(s)g(s)\,\mathrm{d}s \quad \forall g \in C_0^\infty(\mathbb{R}) \tag{3.6}$$

and

$$\tau_{\hat{f}} G(\omega) = G\left(\omega - \int_0^{\cdot} \hat{f}(s)\,\mathrm{d}s\right).$$

Proof. By [Duncan et al. (2000), Theorem 3.1] it suffices to show the result in the case when

$$G(\omega) = \exp^{\diamond}\left(\int_{\mathbb{R}} g(t)\,\mathrm{d}B^{(H)}(t)\right) = \exp^{\diamond}\langle \omega, g\rangle,$$

where g is deterministic and $\|g\|_\phi < \infty$. In this case we have

$$\begin{aligned} F \diamond G &= \exp^{\diamond}\left(\int_{\mathbb{R}} [f(t) + g(t)]\,\mathrm{d}B^{(H)}(t)\right) \\ &= \exp\left(\int_{\mathbb{R}} [f(t) + g(t)]\,\mathrm{d}B^{(H)}(t) - \frac{1}{2}\|f\|_\phi^2 - \frac{1}{2}\|g\|_\phi^2 - (f,g)_\phi\right), \end{aligned}$$

where

$$(f,g)_\phi = \int_{\mathbb{R}^2} f(s)g(t)\phi(s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

But

$$\begin{aligned}\tau_{\hat{f}} G &= \exp^\diamond\left(\int_{\mathbb{R}} g(t)\,\mathrm{d}B^{(H)}(t) - \int_{\mathbb{R}} \hat{f}(t)g(t)\,\mathrm{d}t\right)\\ &= \exp^\diamond\left(\int_{\mathbb{R}} g(t)\,\mathrm{d}B^{(H)}(t) - (f,g)_\phi\right).\end{aligned}$$

Hence

$$\begin{aligned}F\tau_{\hat{f}} G &= \exp\left(\int_{\mathbb{R}} f(t)\,\mathrm{d}B^{(H)}(t) - \frac{1}{2}\|f\|_\phi^2 + \int_{\mathbb{R}} g(t)\,\mathrm{d}B^{(H)}(t) - \frac{1}{2}\|g\|_\phi^2 - (f,g)_\phi\right)\\ &= F \diamond G. \qquad \square\end{aligned}$$

We now return to Eq. (3.3). First let us solve the equation when $b = 0$ and with initial value $X(0)$ given. Namely, let us consider

$$\mathrm{d}X(t) = -\sigma_t X(t)\,\mathrm{d}B^{(H)}(t), \quad X(0) \quad \text{given.} \tag{3.7}$$

With the notion of Wick product, this equation can be written (see [Hu and Øksendal (1999), Definition 3.11])

$$\dot{X}(t) = -\sigma_t X(t) \diamond W^{(H)}(t), \tag{3.8}$$

where $W^{(H)} = \dot{B}^{(H)}$ is the fractional white noise. Using the Wick calculus, we obtain

$$\begin{aligned}X(t) &= X(0) \diamond J_\sigma(t)\\ &:= X(0) \diamond \exp^\diamond\left(-\int_0^t \sigma_s W^{(H)}(s)\,\mathrm{d}s\right)\\ &= X(0) \diamond \exp\left(-\int_0^t \sigma_s\,\mathrm{d}B^{(H)}(s) - \frac{1}{2}\|\sigma\|_{\phi,t}^2\right),\end{aligned} \tag{3.9}$$

where

$$\|\sigma\|_{\phi,t}^2 := \int_0^t \int_0^t \sigma_u \sigma_v \phi(u,v)\,\mathrm{d}u\,\mathrm{d}v. \tag{3.10}$$

To solve Eq. (3.4) we let

$$Y_t := X(t) \diamond J_\sigma(t). \tag{3.11}$$

This means

$$X(t) = Y_t \diamond \hat{J}_\sigma(t), \tag{3.12}$$

where

$$\hat{J}_\sigma(t) = J_{-\sigma}(t) = \exp\left(\int_0^t \sigma_s \, \mathrm{d}B^{(H)}(s) - \frac{1}{2} \|\sigma\|^2_{\phi,t} \right). \tag{3.13}$$

Thus we have

$$\begin{aligned}
\frac{\mathrm{d}Y_t}{\mathrm{d}t} &= \frac{\mathrm{d}X(t)}{\mathrm{d}t} \diamond J_\sigma(t) + X(t) \diamond \frac{\mathrm{d}J_\sigma(t)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}X(t)}{\mathrm{d}t} \diamond J_\sigma(t) - \sigma_t J_\sigma(t) \diamond X(t) \diamond W^{(H)}(t) \\
&= J_\sigma(t) \diamond b(t, X(t), \omega) \\
&= J_\sigma(t) b\left(t, \tau_{-\hat{\sigma}} X(t), \omega + \int_0^{\cdot} \hat{\sigma}(s) \, \mathrm{d}s\right),
\end{aligned}$$

where

$$\int_{\mathbb{R}^2} \sigma_s g(t) \phi(s,t) \, \mathrm{d}s \, \mathrm{d}t = \int_{\mathbb{R}} \hat{\sigma}_s g(s) \, \mathrm{d}s \quad \forall g \in C_0^\infty(\mathbb{R}). \tag{3.14}$$

We are going to relate $\tau_{\hat{\sigma}} X(t)$ to Y_t.

$$\begin{aligned}
\tau_{-\hat{\sigma}} X_t(t,\omega) &= \tau_{-\hat{\sigma}} [J_{-\sigma}(t) \sigma \diamond Y_t(t,\omega)] \\
&= \tau_{-\hat{\sigma}} [J_{-\sigma}(t) \tau_{\hat{\sigma}} Y_t] \\
&= \tau_{-\hat{\sigma}} J_{-\sigma}(t) Y_t.
\end{aligned}$$

Since $\tau_{-\hat{\sigma}} J_{-\sigma}(t) = [J_{-\hat{\sigma}}(t)]^{-1}$, we obtain an equation equivalent to (3.4) for Y_t:

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = J_{-\sigma}(t) b\left(t, [J_{-\sigma}(t)]^{-1} Y_t, \omega + \int_0^{\cdot} \hat{\sigma}(s) \, \mathrm{d}s\right). \tag{3.15}$$

This is a deterministic equation. The initial value $X(0)$ is equivalent to initial value $Y_0 = X(0) \diamond J_{-\sigma}(0) = X(0)$. Thus we can solve the quasi-linear equation with given initial value.

The terminal value $X(T)$ can also be transformed into the terminal value on $Y(T) = X(T) \diamond J_{-\sigma}(T)$. Thus the equation with given terminal value can be solved in a similar way. Note, however, that in this case the solution need not be $\mathscr{F}_{\cdot}^{(H)}$-adapted (see the next section).

Example 3.1. In Eq. (3.4) let us consider the case $b(t,x) = b_t x$ for some deterministic locally bounded function b_t of t. This means that we are considering the linear stochastic differential equation

$$\mathrm{d}X(t) = b_t X(t) \, \mathrm{d}t + \sigma_t X(t) \, \mathrm{d}B^{(H)}(t). \tag{3.16}$$

In this case it is easy to see that Eq. (3.15) satisfied by Y is

$$\dot{Y}_t = b(t) Y_t.$$

When the initial value is $Y(0)=x$ (constant), $x\in\mathbb{R}$, then

$$Y_t = x\mathrm{e}^{\int_0^t b(s)\,\mathrm{d}s}.$$

Thus the solution of (3.16) with $X(0)=x$ can be expressed as

$$\begin{aligned}X(t) &= Y(t)\diamond J_{-\sigma}(t)\\ &= x\exp\left\{\int_0^t b(s)\,\mathrm{d}s + \int_0^t \sigma_s\,\mathrm{d}B^{(H)}(s) - \frac{1}{2}\|\sigma\|^2_{\phi,t}\right\}.\end{aligned} \tag{3.17}$$

If we assume the terminal value $X(T)$ given, then

$$\begin{aligned}Y(t) &= Y(T)\mathrm{e}^{\int_t^T b(s)\,\mathrm{d}s}\\ &= X(T)\diamond J_\sigma(T)\mathrm{e}^{\int_t^T b(s)\,\mathrm{d}s}.\end{aligned}$$

Hence

$$\begin{aligned}X(t) = Y(t)\diamond J_{-\sigma}(t) = X(T)\diamond\exp\left\{\int_t^T b(s)\,\mathrm{d}s\right.\\ \left. - \int_t^T \sigma_s\,\mathrm{d}B^{(H)}(s) - \frac{1}{2}\int_t^T\int_t^T \sigma(u)\sigma(v)\phi(u,v)\,\mathrm{d}u\,\mathrm{d}v\right\}.\end{aligned} \tag{3.18}$$

4. Fractional backward stochastic differential equations

Let $b:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a given function and let $F:\Omega\to\mathbb{R}$ be a given $\mathscr{F}_T^{(H)}$-measurable random variable, where $T>0$ is a constant. Consider the problem of finding $\mathscr{F}^{(H)}$*-adapted processes* $p(t)$, $q(t)$ such that

$$\mathrm{d}p(t) = b(t,p(t),q(t))\,\mathrm{d}t + q(t)\,\mathrm{d}B^{(H)}(t);\quad t\in[0,T], \tag{4.1}$$

$$P(T) = F \quad \text{a.s.} \tag{4.2}$$

This is a fractional backward stochastic differential equation (FBSDE) in the two unknown processes $p(t)$ and $q(t)$. We will not discuss general theory for such equations here, but settle with a solution in a linear variant of (4.1)–(4.2), namely

$$\mathrm{d}p(t) = [\alpha(t) + b_t p(t) + c_t q(t)]\mathrm{d}t + q(t)\,\mathrm{d}B^{(H)}(t);\quad t\in[0,T], \tag{4.3}$$

$$P(T) = F \quad \text{a.s.,} \tag{4.4}$$

where b_t and c_t are given continuous deterministic functions and $\alpha(t)=\alpha(t,\omega)$ is a given $\mathscr{F}^{(H)}$-adapted process s.t. $\int_0^T |\alpha(t,\omega)|\,\mathrm{d}t<\infty$ a.s.

To solve (4.3)–(4.4) we proceed as follows: By the fractional Girsanov theorem (see e.g. [Hu and Øksendal (1999), Theorem 3.18]) we can rewrite (4.3) as

$$\mathrm{d}p(t) = [\alpha(t) + b_t p(t)]\mathrm{d}t + q(t)\,\mathrm{d}\hat{B}^{(H)}(t),\quad t\in[0,T], \tag{4.5}$$

where

$$\hat{B}^{(H)}(t) = B^{(H)}(t) + \int_0^t c_s \,\mathrm{d}s \tag{4.6}$$

is a fractional Brownian motion (with Hurst parameter H) under the new probability measure $\hat{\mu}$ on $\mathscr{F}_T^{(H)}$ defined by

$$\frac{\mathrm{d}\hat{\mu}(\omega)}{\mathrm{d}\mu(\omega)} = \exp^{\diamond}\{-\langle\omega,\hat{c}\rangle\} = \exp\left\{-\int_0^T \hat{c}(s)\,\mathrm{d}B^{(H)}(s) - \frac{1}{2}\|\hat{c}\|_\phi^2\right\}, \tag{4.7}$$

where $\hat{c} = \hat{c}_t$ is the continuous function with $\mathrm{supp}(\hat{c}) \subset [0,T]$ satisfying

$$\int_0^T \hat{c}_s \phi(s,t)\,\mathrm{d}s = c_t, \quad 0 \leqslant t \leqslant T, \tag{4.8}$$

and

$$\|\hat{c}\|_\phi^2 = \int_0^T \int_0^T \hat{c}(s)\hat{c}(t)\phi(s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

If we multiply (4.5) with the integrating factor

$$\beta_t := \exp\left(-\int_0^t b_s\,\mathrm{d}s\right),$$

we get

$$\mathrm{d}(\beta_s p(s)) = \beta_s \alpha(s)\,\mathrm{d}s + \beta_s q(s)\,\mathrm{d}\hat{B}^{(H)}(s), \tag{4.9}$$

or by integrating (4.9) from $s = t$ to T,

$$\beta_T F = \beta_t p(t) + \int_t^T \beta_s \alpha(s)\,\mathrm{d}s + \int_t^T \beta_s q(s)\,\mathrm{d}\hat{B}^{(H)}(s). \tag{4.10}$$

Assume from now on that

$$\|\alpha\|^2_{\mathscr{L}_\phi^{1,2}[0,T]} := \mathbb{E}_{\hat{\mu}}\left[\int_{[0,T]\times[0,T]} \alpha(s)\alpha(t)\phi(s,t)\,\mathrm{d}s\,\mathrm{d}t + \left(\int_0^T \hat{D}_s^\phi \alpha(s)\,\mathrm{d}s\right)^2\right] < \infty. \tag{4.11}$$

By the fractional Itô isometry (see Duncan et al., 2000, Theorem 3.7 or Hu et al., 2000b, (1.10)) applied to $\hat{B}$, $\hat{\mu}$ we then have

$$\mathbb{E}_{\hat{\mu}}\left[\left(\int_0^T \alpha(s)\,\mathrm{d}\hat{B}^{(H)}(s)\right)^2\right] = \|\alpha\|^2_{\mathscr{L}_\phi^{1,2}[0,T]}. \tag{4.12}$$

From now on let us also assume that

$$\mathbb{E}_{\hat{\mu}}[F^2] < \infty. \tag{4.13}$$

We now apply the quasi-conditional expectation operator (see Hu and Øksendal, 1999, Definition 4.9a)

$$\tilde{\mathbb{E}}_{\hat{\mu}}[\,\cdot\,|\mathscr{F}_t^{(H)}]$$

to both sides of (4.10) and get

$$\beta_T \tilde{\mathbb{E}}_{\hat{\mu}}[F|\mathscr{F}_t^{(H)}] = \beta_t p(t) + \int_t^T \beta_s \tilde{\mathbb{E}}_{\hat{\mu}}[\alpha(s)|\mathscr{F}_t^{(H)}]\,\mathrm{d}s. \tag{4.14}$$

Here we have used that $p(t)$ is $\mathscr{F}_t^{(H)}$-measurable, that the filtration $\hat{\mathscr{F}}_t^{(H)}$ generated by $\hat{B}^{(H)}(s)$; $s \leqslant t$ is the same as $\mathscr{F}_t^{(H)}$, and that

$$\tilde{\mathbb{E}}_{\hat{\mu}}\left[\int_t^T f(s,\omega)\,\mathrm{d}\hat{B}^{(H)}(s)|\hat{\mathscr{F}}_t^{(H)}\right] = 0 \quad \text{for all } t \leqslant T \tag{4.15}$$

for all $f \in \hat{\mathscr{L}}_\phi^{1,2}[0,T]$. See [Hu and Øksendal, 1999, Definition 4.9], and [Hu et al., 2000b, Lemma 1.1].

From (4.14) we get the solution

$$p(t) = \exp\left(-\int_t^T b_s\,\mathrm{d}s\right)\tilde{\mathbb{E}}_{\hat{\mu}}[F|\mathscr{F}_t^{(H)}] + \int_t^T \exp\left(-\int_t^s b_r\,\mathrm{d}r\right)\tilde{\mathbb{E}}_{\hat{\mu}}[\alpha(s)|\mathscr{F}_t^{(H)}]\,\mathrm{d}s, \quad t \leqslant T. \tag{4.16}$$

In particular, choosing $t = 0$ we get

$$p(0) = \exp\left(-\int_0^T b_s\,\mathrm{d}s\right)\tilde{\mathbb{E}}_{\hat{\mu}}[F] + \int_0^T \exp\left(-\int_0^s b_r\,\mathrm{d}r\right)\tilde{\mathbb{E}}_{\hat{\mu}}[\alpha(s)]\,\mathrm{d}s. \tag{4.17}$$

Note that $p(0)$ is $\mathscr{F}_0^{(H)}$-measurable and hence a constant. Choosing $t = 0$ in (4.10) we get

$$G = \int_0^T \beta_s q(s)\,\mathrm{d}\hat{B}^{(H)}(s), \tag{4.18}$$

where

$$G = G(\omega) = \beta_T F(\omega) - \int_0^T \beta_s \alpha(s,\omega)\,\mathrm{d}s - p(0) \tag{4.19}$$

with $p(0)$ given by (4.17).

By the fractional Clark–Ocone theorem [Hu and Øksendal (1996), Theorem 4.15b] applied to $(\hat{B}^{(H)}, \hat{\mu})$ we have

$$G = \mathbb{E}_{\hat{\mu}}[G] + \int_0^T \tilde{\mathbb{E}}_{\hat{\mu}}[\hat{D}_s G|\hat{\mathscr{F}}_s^{(H)}]\,\mathrm{d}\hat{B}^{(H)}(s), \tag{4.20}$$

where $\hat{D}$ denotes the Malliavin derivative at s with respect to $\hat{B}^{(H)}(\cdot)$. Comparing (4.18) and (4.20) we see that we can choose

$$q(t) = \exp\left(\int_0^t b_r\,\mathrm{d}r\right)\tilde{\mathbb{E}}_{\hat{\mu}}[\hat{D}_t G|\mathscr{F}_t^{(H)}]. \tag{4.21}$$

We have proved the first part of the following result.

Theorem 4.1. *Assume that* (4.11) *and* (4.13) *hold. Then a solution* $(p(t), q(t))$ *of* (4.3), (4.4) *is given by* (4.16) *and* (4.21). *The solution is unique among all* $\mathscr{F}_{\cdot}^{(H)}$*-adapted processes* $p(\cdot)$, $q(\cdot) \in \hat{\mathscr{L}}_{\phi}^{1,2}[0,T]$.

Proof. It remains to prove uniqueness. The uniqueness of $p(\cdot)$ follows from the way we deduced formula (4.16) from (4.3), (4.4). The uniqueness of q is deduced from (4.18) and (4.20) by the following argument: Substituting (4.20) from (4.18) and using that $\mathbb{E}_{\hat{\mu}}(G) = 0$ we get

$$0 = \int_0^T (\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}}[\hat{D}_s G | \hat{\mathscr{F}}_s^{(H)}])\, \mathrm{d}\hat{B}^{(H)}(s).$$

Hence by the fractional Itô isometry (4.12)

$$0 = \mathbb{E}_{\hat{\mu}}\left[\left\{\int_0^T (\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}}[\hat{D}_s G | \hat{\mathscr{F}}_s^{(H)}])\, \mathrm{d}\hat{B}^{(H)}(s)\right\}^2\right]$$

$$= \|\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}}[\hat{D}_s G | \hat{\mathscr{F}}_s^{(H)}]\|^2_{\hat{\mathscr{L}}_{\phi}^{1,2}[0,T]}$$

from which it follows that

$$\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}}[\hat{D}_s G | \hat{\mathscr{F}}_s^{(H)}] = 0 \quad \text{for } a.a.(s,\omega) \in [0,T] \times \Omega. \qquad \square$$

5. A stochastic maximum principle

We now apply the theory in the previous section to prove a maximum principle for systems driven by fractional Brownian motion. See e.g., Haussman (1986), Peng (1990) and Yong and Zhou (1999) and the references therein for more information about the maximum principle in the classical Brownian motion case.

Suppose $X(t) = X^{(u)}(t)$ is a controlled system of the form

$$\mathrm{d}X(t) = b(t, X(t), u(t))\, \mathrm{d}t + \sigma(t, X(t), u(t))\, \mathrm{d}B^{(H)}(t); \quad X(0) = x \in \mathbb{R}^n, \tag{5.1}$$

where $b : [0.T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$ are given C^1 functions. The control process $u(\cdot) : [0,T] \times \Omega \to U \subset \mathbb{R}^k$ is assumed to be $\mathscr{F}^{(H)}$-adapted. U is a given closed convex set in $\mathbb{R}^k$.

Let $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}^N$ be given C^1 functions and consider a *performance functional* $J(u)$ of the form

$$J(u) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t))\, \mathrm{d}t + g(X(T))\right] \tag{5.2}$$

and a *terminal condition* given by

$$\mathbb{E}[G(X(T))] = 0. \tag{5.3}$$

Let $\mathscr{A}$ denote the set of all $\mathscr{F}_t^{(H)}$-adapted processes $u:[0,T]\times\Omega\to U$ such that $X^{(u)}(t)$ exists and does not explode in $[0,T]$ and

$$E\left[\int_0^T |f(t,X(t),u(t))|\mathrm{d}t + g^-(X(T)) + G^-(X(T))\right] < \infty \tag{5.4}$$

where $y^- = \max(0,y)$ for $y\in\mathbb{R}$, and such that (5.3) holds. If $u\in\mathscr{A}$ and $X^{(u)}(t)$ is the corresponding state process we call $(u,X^{(u)})$ an *admissible pair*. Consider the problem to find J^* and $u^*\in\mathscr{A}$ such that

$$J^* = \sup\{J(u); u\in\mathscr{A}\} = J(u^*). \tag{5.5}$$

If such $u^*\in\mathscr{A}$ exists, then u^* is called an *optimal control* and (u^*,X^*), where $X^* = X^{u^*}$, is called an *optimal pair*.

Let $\mathscr{R}^{n\times m}$ be the set of continuous function from $[0,T]$ into $\mathbb{R}^{n\times m}$. Define the Hamiltonian $H:[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathscr{R}^{n\times m}\to\mathbb{R}$ by

$$\begin{aligned} H(t,x,u,p,q(\cdot)) &= f(t,x,u) + b(t,x,u)^{\mathrm{T}}p \\ &\quad + \sum_{i=1}^n\sum_{k=1}^m \sigma_{ik}(t,x,u)\int_0^T q_{ik}(s)\phi_{H_k}(s,t)\,\mathrm{d}s. \end{aligned} \tag{5.6}$$

Consider the following fractional stochastic backward differential equation in the pair of unknown $\mathscr{F}_t^{(H)}$-adapted processes $p(t)\in\mathbb{R}^n$, $q(t)\in\mathbb{R}^{n\times m}$, called the *adjoint processes*:

$$\begin{aligned} \mathrm{d}p(t) &= -H_x(t,X(t),u(t),p(t),q(\cdot))\,\mathrm{d}t + q(t)\,\mathrm{d}B^{(H)}(t); \quad t\in[0,T] \\ p(T) &= g_x(X(T)) + \lambda^{\mathrm{T}}G_x(X(T)). \end{aligned} \tag{5.7}$$

where $H_x = \nabla_x H = (\frac{\partial H}{\partial x_1},\ldots,\frac{\partial H}{\partial x_n})^{\mathrm{T}}$ is the gradient of H with respect to x and similarly with g_x and G_x. $X(t) = X^{(u)}(t)$ is the process obtained by using the control $u\in\mathscr{A}$ and $\lambda\in\mathbb{R}^n_+$ is a constant. Eq. (5.6) is called the adjoint equation and $p(t)$ is sometimes interpreted as the *shadow price* (of a resource).

Theorem 5.1 (The fractional stochastic maximum principle). *Suppose $\hat{u}\in\mathscr{A}$ and put $\hat{X} = X^{(\hat{u})}$. Suppose there exists a solution $\hat{p}(t),\hat{q}(t)$ of the corresponding adjoint Eq.* (5.7) *for some $\lambda\in\mathbb{R}^n_+$ and such that the following*, (5.8)–(5.11), *hold*

$$X^{(u)}(t)\hat{q}(t)\in\mathscr{L}_\phi^{1,2} \quad \textit{and} \quad \hat{p}^{\mathrm{T}}(t)\sigma(t,X^{(u)}(t),u(t))\in\mathscr{L}_\phi^{1,2} \ \textit{for all } u\in\mathscr{A}, \tag{5.8}$$

$$H(t,\cdot,\cdot,\hat{p}(t),\hat{q}(t)),\ g(\cdot) \quad \textit{and} \quad G(\cdot) \textit{ are concave, for all } t\in[0,T], \tag{5.9}$$

$$H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(\cdot)) = \max_{v\in U} H(t,\hat{X}(t),v,\hat{p}(t),\hat{q}(\cdot)), \tag{5.10}$$

$$\begin{aligned} \Delta_4 := \mathbb{E}\Bigg[\sum_{i=1}^n\sum_{j,k=1}^m &\left(\int_0^T D_{j,t}^{\phi_j}\{\sigma_{ik}(t,X(t),u(t))\right. \\ &\left. - \sigma_{ik}(t,\hat{X}(t),\hat{u}(t))\}\,\mathrm{d}t\right)\left(\int_0^T D_{k,t}^{\phi_k}\hat{q}_{ij}(t)\,\mathrm{d}t\right)\Bigg] \leqslant 0 \quad \textit{for all } u\in\mathscr{A}. \end{aligned} \tag{5.11}$$

Then if $\lambda \in \mathbb{R}_+^n$ *is such that* $(\hat{u}, \hat{X})$ *is admissible* (*in particular*, (5.3) *holds*), *the pair* $(\hat{u}, \hat{X})$ *is an optimal pair for problem* (5.5).

Proof. We first give a proof in the case when $G(x) = 0$, i.e. when there is no terminal condition.

With $(\hat{u}, \hat{X})$ as above consider

$$\begin{aligned}
\Delta :=\ & \mathbb{E}\left[\int_0^T f(t,\hat{X}(t),\hat{u}(t))\,\mathrm{d}t - \int_0^T f(t,X(t),u(t))\,\mathrm{d}t\right] \\
=\ & \mathbb{E}\left[\int_0^T H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(\cdot))\,\mathrm{d}t - \int_0^T H(t,X(t),u(t),\hat{p}(t),\hat{q}(\cdot))\,\mathrm{d}t\right] \\
& -\mathbb{E}\left[\int_0^T \{b(t,\hat{X}(t),\hat{u}(t))\}^{\mathrm{T}}\hat{p}(t)\,\mathrm{d}t - \int_0^T b(t,X(t),u(t))^{\mathrm{T}}\hat{p}(t)\,\mathrm{d}t\right] \\
& -\mathbb{E}\left[\int_0^T\int_0^T \sum_{i=1}^n\sum_{k=1}^m \{\sigma_{ik}(s,\hat{X}(s),\hat{u}(s))\right. \\
& \left. -\sigma_{ik}(s,X(s),u(s))\}\hat{q}_{ik}(t)\phi_{H_k}(s,t)\,\mathrm{d}s\,\mathrm{d}t\right] \\
=:\ & \Delta_1 + \Delta_2 + \Delta_3.
\end{aligned} \tag{5.12}$$

Since $(x,u) \to H(x,u) = H(t,x,u,p,q(\cdot))$ is concave we have

$$H(x,u) - H(\hat{x},\hat{u}) \leqslant H_x(\hat{x},\hat{u})(x-\hat{x}) + H_u(\hat{x},\hat{u})(u-\hat{u})$$

for all (x,u), $(\hat{x},\hat{u})$. Since $v \to H(\hat{X}(t),v)$ is maximal at $v = \hat{u}(t)$ we have

$$H_u(\hat{x},\hat{u})(u(t)-\hat{u}(t)) \leqslant 0 \quad \forall t.$$

Therefore

$$\begin{aligned}
\Delta_1 &\geqslant \mathbb{E}\left[\int_0^T -H_x(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(\cdot))\cdot(X(t)-\hat{X}(t))\,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T (X(t)-\hat{X}(t))^{\mathrm{T}}\,\mathrm{d}\hat{p}(t) - \int_0^T (X(t)-\hat{X}(t))^{\mathrm{T}}\hat{q}(t)\,\mathrm{d}B^{(H)}(t)\right].
\end{aligned}$$

Since $\mathbb{E}[\int_0^T (X(t)-\hat{X}(t))^{\mathrm{T}}\hat{q}(t)\,\mathrm{d}B^{(H)}(t)] = 0$ by (2.7), this gives

$$\Delta_1 \geqslant \mathbb{E}\left[\int_0^T (X(t)-\hat{X}(t))^{\mathrm{T}}\,\mathrm{d}\hat{p}(t)\right]. \tag{5.13}$$

By (5.1) we have

$$\begin{aligned}
\Delta_2 &= -\mathbb{E}\left[\int_0^T \{b(t,\hat{X}(t),\hat{u}(t)) - b(t,X(t),u(t))\}\,\hat{p}(t)\,\mathrm{d}t\right] \\
&= -\mathbb{E}\left[\int_0^T \hat{p}(t)(\mathrm{d}\hat{X}(t) - \mathrm{d}X(t))\right]
\end{aligned}$$

$$- \mathbb{E}\left[\int_0^T \hat{p}(t)^{\mathrm{T}}\{\sigma(t,\hat{X}(t),\hat{u}(t)) - \sigma(t,X(t),u(t))\}\,\mathrm{d}B^{(H)}(t)\right]$$

$$= \mathbb{E}\left[\int_0^T \hat{p}(t)(\mathrm{d}X(t) - \mathrm{d}\hat{X}(t))\right]. \tag{5.14}$$

Finally, since g is concave we have

$$g(X(T)) - g(\hat{X}(T)) \leqslant g_x(\hat{X}(T))(X(T) - \hat{X}(T)). \tag{5.15}$$

Combining (5.12)–(5.15) with Corollary 2.5 we get, using (5.2), (5.7) and (5.11),

$$\begin{aligned}
J(\hat{u}) - J(u) &= \Delta + \mathbb{E}[g(\hat{X}(T)) - g(X(T))] \\
&\geqslant \Delta + \mathbb{E}[g_x(\hat{X}(T))(\hat{X}(T) - X(T))] \\
&\geqslant \Delta - \mathbb{E}[\hat{p}(T)(X(T) - \hat{X}(T))] \\
&= \Delta - \left\{\mathbb{E}\left[\int_0^T (X(t) - \hat{X}(t))\,\mathrm{d}\hat{p}(t)\right] + \mathbb{E}\left[\int_0^T \hat{p}(t)(\mathrm{d}X(t) - \mathrm{d}\hat{X}(t))\right]\right. \\
&\quad + \mathbb{E}\left[\int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \{\sigma_{ik}(s,X(s),u(s)) - \sigma_{ik}(s,\hat{X}(s),\hat{u}(s))\}\right. \\
&\quad \left. \times\, \hat{q}_{ik}(t)\phi_{H_k}(s,t)\,\mathrm{d}s\,\mathrm{d}t\right] \\
&\quad + \mathbb{E}\left[\sum_{i=1}^n \sum_{j,k=1}^m \left(\int_0^T D_{j,t}^{\phi_j}\{\sigma_{ik}(t,X(t),u(t)) - \sigma_{ik}(t,\hat{X}(t),\hat{u}(t))\}\,\mathrm{d}t\right)\right. \\
&\quad \left.\left. \times \left(\int_0^T D_{k,t}^{\phi_k}\hat{q}_{ij}(t)\right)\right]\right\} \\
&\geqslant \Delta - (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \geqslant 0.
\end{aligned}$$

This shows that $J(\hat{u})$ is maximal among all admissible pairs $(u(\cdot), X(\cdot))$.

This completes the proof in the case with no terminal conditions ($G = 0$). Finally consider the general case with $G \neq 0$. Suppose that for some $\lambda_0 \in \mathbb{R}_+^n$ there exists $\hat{u}_{\lambda_0}$ satisfying (5.8)–(5.11). Then by the above argument we know that if we put

$$J_{\lambda_0}(u) = \mathbb{E}\left[\int_0^T f(t,X(t),u(t))\,\mathrm{d}t + g(X(T)) + \lambda_0^{\mathrm{T}} G(X(T))\right]$$

then $J_{\lambda_0}(\hat{u}_0) \geqslant J_{\lambda_0}(u)$ for all controls u (without terminal condition). If λ_0 is such that $\hat{u}_{\lambda_0}$ satisfies the terminal condition (i.e. $\hat{u}_{\lambda_0} \in \mathscr{A}$) and u is another control in $\mathscr{A}$ then

$$J(\hat{u}_{\lambda_0}) = J_{\lambda_0}(\hat{u}_{\lambda_0}) \geqslant J_{\lambda_0}(u) = J(u)$$

and hence $\hat{u}_{\lambda_0} \in \mathscr{A}$ maximizes $J(u)$ over all $u \in \mathscr{A}$. □

Corollary 5.2. *Let* $\hat{u} \in \mathscr{A}$, $\hat{X} = X^{(\hat{u})}$ *and* $(\hat{p}(t), \hat{q}(t))$ *be as in Theorem* 5.1. *Assume that* (5.8), (5.9) *and* (5.10) *hold, and that condition* (5.11) *is replaced by the condition*

$$\hat{q}(\cdot) \text{ or } \sigma(\cdot, \hat{X}(\cdot), \hat{u}(\cdot)) \text{ is deterministic.} \tag{5.16}$$

Then if $\lambda \in \mathbb{R}^n_+$ *is such that* $(\hat{u}, \hat{X})$ *is admissible, the pair* $(\hat{u}, \hat{X})$ *is an optimal pair for problem* (5.5).

6. A minimal variance hedging problem

To illustrate our main result, we use it to solve the following problem from mathematical finance.

Consider a financial market driven by two independent fractional Brownian motions $B_1(t) = B_1^{(H_1)}(t)$ and $B_2(t) = B^{(H_2)}(t)$, with $\frac{1}{2} < H_i < 1$, $i = 1, 2$, as follows:

$$\text{(Bond price)} \quad \mathrm{d}S_0(t) = 0, \quad S_0(0) = 1, \tag{6.1}$$

$$\text{(Price of stock 1)} \quad \mathrm{d}S_1(t) = \mathrm{d}B_1(t), \quad S_1(0) = s_1, \tag{6.2}$$

$$\text{(Price of stock2)} \quad \mathrm{d}S_2(t) = \mathrm{d}B_1(t) + \mathrm{d}B_2(t), \quad S_2(0) = s_2. \tag{6.3}$$

If $\theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t)) \in \mathbb{R}^3$ is a *portfolio* (giving the number of units of the bond, stock 1 and stock 2, respectively, held at time t) then the corresponding *value process* is

$$V^\theta(t) = \theta(t) S(t) = \sum_{i=0}^{2} \theta_i(t) S_i(t). \tag{6.4}$$

The portfolio is called *self-financing* if

$$\mathrm{d}V^\theta(t) = \theta(t)\, \mathrm{d}S(t) = \theta_1(t)\, \mathrm{d}B_1(t) + \theta_2(t)(\mathrm{d}B_1(t) + \mathrm{d}B_2(t)). \tag{6.5}$$

This market is called *complete* if any bounded $\mathscr{F}_T^{(H)}$-measurable random variable F can be *hedged* (or *replicated*), in the sense that there exists a (self-financing) portfolio $\theta(t)$ and an initial value $z \in \mathbb{R}$ such that

$$F(\omega) = z + \int_0^T \theta(t)\, \mathrm{d}S(t) \quad \text{for a.a. } \omega. \tag{6.6}$$

(See Hu and Øksendal (1999) and Wallner (2001) for a general discussion about this.)

Let us now assume that we are not allowed to trade in stock 1, i.e. we must have $\theta_1(t) \equiv 0$. How close to, say, $F(\omega) = B_1(T, \omega)$ can we get if we must hedge under this constraint?

If we put $\theta_2(t) = u(t)$ and interpret "close" as having a small $L^2(\mu)$ distance to F, then the problem can be stated as follows.

Find $z \in \mathbb{R}$ and admissible $u(t, \omega)$ such that

$$\begin{aligned} J(z,u) &:= \mathbb{E}\left[\left\{B_1(T) - \left(z + \int_0^T u(t)(\mathrm{d}B_1(t) + \mathrm{d}B_2(t))\right)\right\}^2\right] \\ &= z^2 + \mathbb{E}\left[\left\{\int_0^T (u(t) - 1)\, \mathrm{d}B_1(t) + \int_0^T u(t)\, \mathrm{d}B_2(t)\right\}^2\right] \end{aligned} \tag{6.7}$$

is minimal. We see immediately that it is optimal to choose $z = 0$, so it remains to minimize over $u(t) = u(t, \omega)$ the functional

$$J(u) := \mathbb{E}\left[\left\{\int_0^T (u(t)-1)\,\mathrm{d}B_1(t) + \int_0^T u(t)\,\mathrm{d}B_2(t)\right\}^2\right]. \tag{6.8}$$

If we apply the fractional Itô isometry (2.13) we get, after some simplifications

$$\begin{aligned} J(u) = \mathbb{E}\Bigg[& \int_0^T \int_0^T \{(u(s)-1)(u(t)-1)\phi_1(s,t) + u(s)u(t)\phi_2(s,t)\}\,\mathrm{d}s\,\mathrm{d}t \\ & + \left(\int_0^T \{D_{1,t}^{\phi} u(t) - D_{2,t}^{\phi} u(t)\}\,\mathrm{d}t\right)^2\Bigg]. \end{aligned} \tag{6.9}$$

However, it is difficult to see from this what the minimizing $u(t)$ is.

To approach this problem by using the fractional maximum principle, we define the state process $X(t)$ by

$$\mathrm{d}X(t) = (u(t)-1)\,\mathrm{d}B_1(t) + u(t)\,\mathrm{d}B_2(t). \tag{6.10}$$

Then the problem is equivalent to maximizing

$$J_1(u) := \mathbb{E}[-\tfrac{1}{2}X^2(T)]. \tag{6.11}$$

The Hamiltonian for this problem is

$$\begin{aligned} H(t,x,u,p,q(\cdot)) &= (u-1)\int_0^T q_1(s)\phi_1(s,t)\,\mathrm{d}s + u\int_0^T q_2(s)\phi_2(s,t)\,\mathrm{d}s \\ &= (u-1)\int_0^T q_1(s)\phi_1(s,t)\,\mathrm{d}s + u\int_0^T q_2(s)\phi_2(s,t)\,\mathrm{d}s \\ &= u\left[\int_0^T q_1(s)\phi_1(s,t)\,\mathrm{d}s + \int_0^T q_2(s)\phi_2(s,t)\,\mathrm{d}s\right] \\ &\quad - \int_0^T q_1(s)\phi_1(s,t)\,\mathrm{d}s. \end{aligned} \tag{6.12}$$

The adjoint equation is

$$\mathrm{d}p(t) = q_1(t)\,\mathrm{d}B_1(t) + q_2(t)\,\mathrm{d}B_2(t), \quad t < T, \tag{6.13}$$

$$p(T) = -X(T). \tag{6.14}$$

Comparing with (6.10) we see that this equation has the solution

$$q_1(t) = 1 - u(t), \quad q_2 = -u_2(t), \quad p(t) = -X(t), \quad t \leqslant T. \tag{6.15}$$

Let $\hat{u}(t)$ be an optimal control candidate. Then by (6.12)

$$\begin{aligned} H(t,\hat{X}(t),v,\hat{p}(t),\hat{q}(\cdot)) &= v\left[\int_0^T \hat{q}_1(s)\phi_1(s,t)\,\mathrm{d}s + \int_0^T \hat{q}_2(s)\phi_2(s,t)\,\mathrm{d}s\right] \\ &\quad - \int_0^T \hat{q}_1(s)\phi_1(s,t)\,\mathrm{d}s \\ &= v\left[\int_0^T (1-\hat{u}(t))\phi_1(s,t)\,\mathrm{d}s - \int_0^T \hat{u}(s)\phi_2(s,t)\,\mathrm{d}s\right] \\ &\quad - \int_0^T \hat{q}_1(s)\phi_1(s,t)\,\mathrm{d}s. \end{aligned} \tag{6.16}$$

The maximum principle requires that the maximum of this expression is attained at $v=\hat{u}(t)$. However, this is an affine function of v, so it is natural to guess that the coefficient of v must be 0, i.e.

$$\int_0^T (1-\hat{u}(s))\phi_1(s,t)\,\mathrm{d}s - \int_0^T \hat{u}(s)\phi_2(s,t)\,\mathrm{d}s = 0$$

which gives

$$\int_0^T \hat{u}(s)(\phi_1(s,t)+\phi_2(s,t))\,\mathrm{d}s = \int_0^T \phi_1(s,t)\,\mathrm{d}s. \tag{6.17}$$

This is a symmetric Fredholm integral equation of the first kind and it is known that it has a unique solution $\hat{u}(t) \in L^2[0,T]$. See e.g. [Tricomi (1985), Section 3.15].

This choice of $\hat{u}(t)$ satisfies all the requirements of Theorem 5.1 (in fact, even those of Corollary 5.2) and we can conclude that this $\hat{u}(t)$ is optimal. Thus we have proved.

Theorem 6.1 (Solution of the minimal variance hedging problem). *The minimal value of*

$$J(z,u) = \mathbb{E}\left[\left\{B_1(T) - \left(z + \int_0^T u(t)(\mathrm{d}B_1(t)+\mathrm{d}B_2(t))\right)\right\}^2\right]$$

is attained when $z=0$ *and* $u=\hat{u}(t)$ *satisfies* (6.17). *The corresponding minimal value is*

$$\inf_{z,u} J(z,u) = \int_0^T\int_0^T \{(\hat{u}(s)-1)(\hat{u}(t)-1)\phi_1(s,t) + \hat{u}(s)\hat{u}(t)\phi_2(s,t)\}\,\mathrm{d}s\,\mathrm{d}t.$$

Remark. Note that if $\phi_1=\phi_2$ then $\hat{u}(t)\equiv\frac{1}{2}$, which is the same as the optimal value in the classical Brownian motion case $(H_1=H_2=\frac{1}{2})$.

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